

PBW BASES AND marginally LARGE TABLEAUX IN TYPE D

BEN SALISBURY, ADAM SCHULTZE, AND PETER TINGLEY

ABSTRACT. We give an explicit description of the unique crystal isomorphism between two realizations of $B(\infty)$ in type D : that using marginally large tableaux and that using PBW monomials with respect to one particularly nice reduced expression of the longest word.

1. INTRODUCTION

For any symmetrizable Kac-Moody algebra, the crystal $B(\infty)$ is a combinatorial object that contains information about the corresponding universal enveloping algebra and its integrable highest weight representations. Kashiwara's definition of $B(\infty)$ uses some intricate algebraic constructions, but it can often be realized in quite simple ways. We consider two such realizations.

- (1) The construction using marginally large tableaux from [6].
- (2) The recent construction using bracketing rules on Kostant partitions from [12], which is naturally identified with the algebraic crystal structure on PBW monomials for one particularly nice reduced expression of w_0 .

We give an explicit description of the unique crystal isomorphism between these two realizations (see Theorem 3.1). This is a type D analogue of a type A result that can be found in [3], although the type D situation is a little different. Most notably, the isomorphism is not as “local:” in type A , the map from tableaux to Kostant partitions simply maps each box in the tableau to a root, but in type D one must consider multiple boxes at once. In the final section we give a diagrammatic description of Kostant partitions and the crystal operators on them which mimics the diagrams implicit from the multisegment picture in type A [3, 7, 9, 13].

2. BACKGROUND

Let \mathfrak{g} be the Lie algebra of type D_n with Cartan matrix and Dynkin diagram

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}, \quad \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-2} \\ \alpha_{n-1} \\ \alpha_n \end{array}$$

Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the simple coroots, related by the inner product $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$. Define the fundamental weights $\{\omega_1, \dots, \omega_n\}$ by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. Then the weight lattice is $P = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n$ and the coweight lattice

B.S. was partially supported by CMU Early Career grant #C62847.

A.S. and P.T. were partially supported by NSF grant DMS-1265555.

$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \leq i < k \leq n$
$\beta_{i,k} = \epsilon_i - \epsilon_{k+1},$	$1 \leq i \leq k \leq n-1$
$\gamma_{i,k} = \epsilon_i + \epsilon_k,$	$1 \leq i < k \leq n$

TABLE 2.1. Positive roots of type D_n , expressed both as a linear combination of simple roots and in the canonical realization following [2].

is $P^\vee = \mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_n^\vee$. The Cartan subalgebra \mathfrak{h} is given by $\mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. Also, let Φ denote the roots associated to \mathfrak{g} , with the set of positive roots denoted Φ^+ . The list of positive roots is given in Table 2.1.

Let $B(\infty)$ be the infinity crystal associated to \mathfrak{g} as defined in [8]. This is a countable set along with operators e_i and f_i which roughly correspond to the Chevalley generators of \mathfrak{g} . We don't need the details of the definition of $B(\infty)$, as we just consider two explicitly defined ways to realize it.

2.1. Type D marginally large tableaux.

Definition 2.1. A marginally large tableau of type D_n is an $n-1$ row tableau on the alphabet

$$J(D_n) := \left\{ 1 \prec \cdots \prec n-1 \prec \frac{n}{\bar{n}} \prec \overline{n-1} \prec \cdots \prec \bar{1} \right\}$$

which satisfies the following conditions.

- (1) The first column has entries $1, 2, \dots, n-1$ in that order.
- (2) Entries weakly increase along rows.
- (3) The number of i -boxes in the i th row is exactly one more than the total number of boxes in the $(i+1)$ st row. We call this condition “marginal largeness.”
- (4) Every entry in the i th row is $\preceq \bar{i}$.
- (5) The entries n and \bar{n} do not appear in the same row.

Denote by $\mathcal{T}(\infty)$ the set of marginally large tableaux.

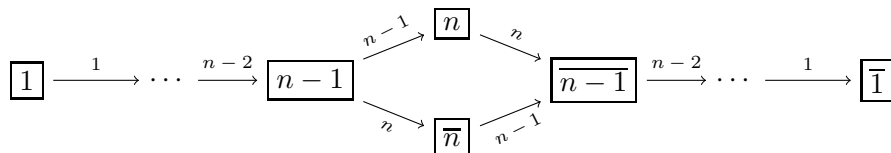
Example 2.2. In type D_4 , the elements of $\mathcal{T}(\infty)$ all have the form

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 \cdots 1 & 1 \cdots 1 & 1 & 1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 & 1 & 2 \cdots 2 & 3 \cdots 3 & x_1 \cdots x_1 & \overline{3} \cdots \overline{3} & \overline{2} \cdots \overline{2} & \overline{1} \cdots \overline{1} \\ \hline 2 & 2 \cdots 2 & 2 \cdots 2 & 2 & 3 \cdots 3 & x_2 \cdots x_2 & \overline{3} \cdots \overline{3} & \overline{2} \cdots \overline{2} & & & & & & & \\ \hline 3 & x_3 \cdots x_3 & \overline{3} \cdots \overline{3} & & & & & & & & & & & & \\ \hline \end{array},$$

where $x_i \in \{4, \bar{4}\}$ for each $i = 1, 2, 3$. We typically shade the i -boxes in row i , as these are basically placeholders. In particular, the unique element of weight zero is

$$T_\infty = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \\ \hline 3 & & \\ \hline \end{array}.$$

Definition 2.3. Fix a type D_n marginally large tableau. The reading word $\text{read}(T)$ is obtained by reading right to left along rows, starting at the top and working down.



The rightmost uncanceled ‘)’ is the one shown in blue, so e_4 changes the corresponding $\overline{4}$ -box in the third row to a 3-box. To maintain marginal largeness, we must also slide the first row one unit to the left so that we have exactly one more 3-box in the third

row than total number of boxes in the (empty) fourth row:

$$e_4 T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & \overline{3} & \overline{1} & \overline{1} & \overline{1} & \\ \hline 2 & 2 & 2 & 3 & \overline{4} & \overline{3} & \overline{3} & & & & & & & & & \\ \hline 3 & \overline{3} & & & & & & & & & & & & & & \\ \hline \end{array}.$$

Similarly,

$$f_4 T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & \overline{3} & \overline{1} & \overline{1} & \overline{1} & \\ \hline 2 & 2 & 2 & 2 & 2 & 3 & \overline{4} & \overline{3} & \overline{3} & & & & & & & & \\ \hline 3 & \overline{4} & \overline{4} & \overline{3} & & & & & & & & & & & & & \\ \hline \end{array}.$$

We are using the so-called middle-Eastern reading, as defined in [5]. This differs from the original definition of the signature rule for element of $\mathcal{T}(\infty)$ given in [6] which uses the far-Eastern reading. However, the resulting operators are identical.

Proposition 2.7. *The operators e_i and f_i on $\mathcal{T}(\infty)$ defined using the far-Eastern reading and the middle-Eastern reading, respectively, are identical.*

Proof. Fix $T \in \mathcal{T}(\infty)$ and let c_{ij} be the number of j -boxes in row i of T . First assume $1 \leq i \leq n-2$. Then all brackets used in calculating f_i come from rows $1, \dots, i+1$. The brackets corresponding to unshaded boxes come in exactly the same order for the two readings. Thus the only difference between the two bracket orders is at the right end of the sequence, where one has:

$$\begin{aligned} \text{far-Eastern: } & \dots (c_{i,i} - c_{i+1,i+1} + \underbrace{c_{\overline{i+1},i+1}}_{c_{i+1,i+1}}) \dots (), \\ \text{middle-Eastern: } & \dots (c_{i,i} (c_{\overline{i+1},i+1})^{c_{i+1,i+1}}). \end{aligned} \tag{2.8}$$

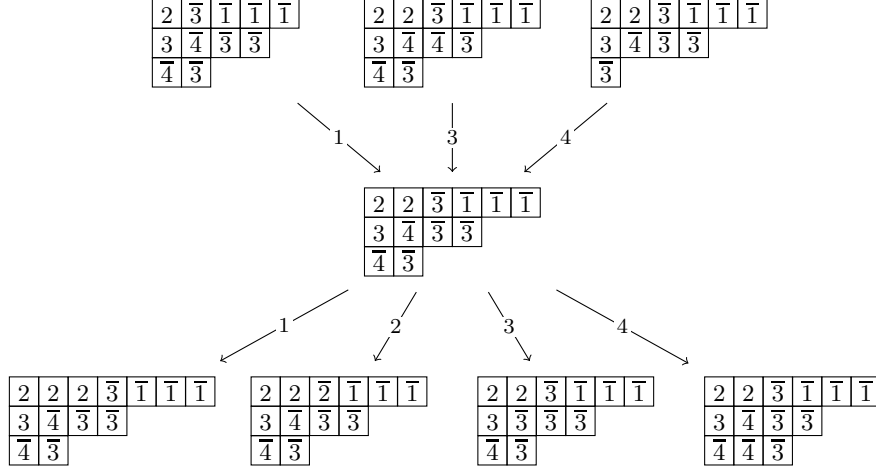
Since $c_{i,i} > c_{i+1,i+1}$, the portions shown each have no uncanceled, ‘),’ and they have the same number of uncanceled ‘(,’ with the first uncanceled ‘(’ corresponding to a shaded i . It follows that the first uncanceled bracket of each type in the two sequences corresponds to a box of the same type (i.e., same content and on same row). Clearly both rules always apply f_i to the leftmost box of a given type, and e_i to the rightmost, so the two rules agree.

The argument for $i = n-1, n$ is similar, and in fact simpler, since the only shaded boxes that are relevant are the shaded $n-1$. ■

Remark 2.9. Unlike in type A , the operators on finite type D tableaux using these two readings are different. They only agree for marginally large tableaux.

Since the shaded boxes of a marginally large tableau are merely placeholders, we sometimes omit them. For a tableau T , consider the *reduced form* of T , which is obtained by removing all shaded boxes and sliding the rows so that the result is left-justified.

Example 2.10. Continuing Example 2.6, we can picture the crystal graph around T using tableaux in reduced form.



2.2. Crystal structure on Kostant partitions. Here we review the crystal structure on Kostant partitions from [12]. As explained there, this is naturally identified with the crystal structure on PBW monomials from, for example, [1, 10] for the reduced expression

$$w_0 = (s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1} s_n s_{n-2}) s_{n-1} s_n.$$

Let \mathcal{R} be the set of symbols $\{(\beta) : \beta \in \Phi^+\}$. Let $\text{Kp}(\infty)$ be the free $\mathbf{Z}_{\geq 0}$ -span of \mathcal{R} . This is the set of *Kostant partitions*. We denote elements of $\text{Kp}(\infty)$ by $\alpha = \sum_{(\beta) \in \mathcal{R}} c_\beta (\beta)$. If $c_\beta \neq 0$, we say that α is *supported* on β and that (β) is a *part* of α .

Definition 2.11. Consider the following subsets of positive roots depending on $i \in I$.

- (1) For $1 \leq i \leq n-1$, define

$$\Phi_i = \{\beta_{k,i-1}, \beta_{k,i} : 1 \leq k \leq i\} \cup \{\gamma_{k,i}, \gamma_{k,i+1} : 1 \leq k \leq i-1\}$$

and order the roots in Φ_i by

$$\beta_{1,i} < \beta_{1,i-1} < \gamma_{1,i} < \gamma_{1,i+1} < \cdots < \beta_{i-1,i} < \beta_{i-1,i-1} < \gamma_{i-1,i} < \gamma_{i-1,i+1} < \beta_{i,i}.$$

- (2) For $i = n$, define

$$\Phi_n = \{\beta_{k,n-2}, \beta_{k,n-1} : 1 \leq k \leq n-2\} \cup \{\gamma_{k,n-1}, \gamma_{k,n} : 1 \leq k \leq n-2\} \cup \{\gamma_{n-1,n}\}$$

and order the roots in Φ_n by

$$\gamma_{1,n} < \beta_{1,n-2} < \gamma_{1,n-1} < \beta_{1,n-1} < \cdots < \gamma_{n-2,n} < \beta_{n-2,n-2} < \gamma_{n-2,n-1} < \beta_{n-2,n-1} < \gamma_{n-1,n}.$$

The *bracketing sequence* $S_i(\alpha)$ consists of, for each $\beta \in \Phi_i$, c_β -many ‘)’ if $\beta - \alpha_i$ is a positive root and c_β -many ‘(’ if $\beta + \alpha_i$ is a positive root, ordered as above. Successively cancel ‘)’-pairs to obtain sequence of the form $) \cdots) (\cdots ($. We call the remaining brackets *uncanceled*.

Definition 2.12. Let $i \in I$ and $\alpha = \sum_{(\beta) \in \mathcal{R}} c_\beta(\beta) \in \text{Kp}(\infty)$.

- Let β be the root corresponding to the rightmost uncanceled ‘)’ in $S_i(\alpha)$. Define

$$e_i \alpha = \alpha - (\beta) + (\beta - \alpha_i).$$

If $\beta = \alpha_i$, we interpret (0) as the additive identity in $\mathbf{Kp}(\infty)$. If no such ‘ i ’ exists, then $e_i \alpha$ is undefined.

- Let γ denote the root corresponding to the leftmost uncanceled '(' in $S_i(\alpha)$. Define,

$$f_i \alpha = \alpha - (\gamma) + (\gamma + \alpha_i).$$

If no such ‘(’ exists, set $f_i\alpha = \alpha + (\alpha_i)$.

- $\text{wt}(\alpha) = - \sum_{\beta \in \Phi^+} c_\beta \beta$.
- $\varepsilon_i(\alpha) = \text{number of '}' \text{ in the bracketing sequence of } \alpha$.
- $\varphi_i(\alpha) = \varepsilon_i(\alpha) + \langle \alpha_i^\vee, \text{wt}(\alpha) \rangle$.

Proposition 2.13 ([12]). *With the operations defined above, $\text{Kp}(\infty)$ realizes $B(\infty)$. ■*

Example 2.14. Let $i = n = 4$ and consider

$$\begin{aligned} \alpha = & 5(\alpha_1) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + 3(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) \\ & + 2(\alpha_2 + \alpha_4) + (\alpha_2 + \alpha_3) + (\alpha_2 + \alpha_3 + \alpha_4) + (\alpha_3) + 2(\alpha_4). \end{aligned}$$

Look at the coefficients c_β of α corresponding to $\beta \in \Phi_4$

$$0\gamma_{1,4} \quad 0\beta_{1,2} \quad \gamma_{1,3} \quad 0\beta_{1,3} \quad 2\gamma_{2,4} \quad 0\beta_{2,2} \quad \gamma_{2,3} \quad \beta_{2,3} \quad 2\gamma_{3,4}$$

Hence, $e_4\alpha = \alpha - (\alpha_4) + (0) = \alpha - (\alpha_4)$ and $f_4\alpha = \alpha + (\alpha_4)$.

3. THE ISOMORPHISM

Theorem 3.1. *The unique crystal isomorphism $\Psi: \mathcal{T}(\infty) \rightarrow \text{Kp}(\infty)$ can be described as follows. For a tableau $T \in \mathcal{T}(\infty)$, let R_1, \dots, R_{n-1} denote the rows of T starting at the top. Set $\Psi(T) = \sum_{j=1}^{n-1} \Psi(R_j)$, where $\Psi(R_j)$ is defined in the following way:*

- (1) if $j \neq n-1$, each \bar{j} is sent to $(\beta_{j,j}) + (\gamma_{j,j+1})$;
- (2) if $j = n-1$, each \bar{j} is sent to $(\beta_{n-1,n-1}) + (\gamma_{n-1,n})$;
- (3) each pair k, \bar{k} , where $k \neq n-1$, maps to $(\beta_{j,k}) + (\gamma_{j,k+1})$;
- (4) each pair $n-1, \overline{n-1}$ maps to $(\beta_{j,n-1}) + (\gamma_{j,n})$;
- (5) each remaining $k \in \{j, j+1, \dots, n\}$ is sent to $(\beta_{j,k-1})$;
- (6) each remaining $\bar{k} \in \{\overline{n}, \overline{n-1}, \dots, \overline{j+1}\}$ is sent to $(\gamma_{j,k})$.

Example 3.2. Let $n = 4$ and

[illegible]

Then

$$\begin{aligned}\Psi(R_1) &= 3((\beta_{1,1}) + (\gamma_{1,2})) + (\gamma_{1,3}) + 2(\beta_{1,1}), \\ \Psi(R_2) &= ((\beta_{2,3}) + (\gamma_{2,4})) + (\gamma_{2,3}) + (\beta_{2,4}), \\ \Psi(R_3) &= ((\beta_{3,3}) + (\gamma_{3,4})) + (\gamma_{3,4}),\end{aligned}$$

so

$$\Psi(T) = 5(\beta_{1,1}) + (\gamma_{1,3}) + 3(\gamma_{1,2}) + 2(\gamma_{2,4}) + (\beta_{2,3}) + (\gamma_{2,3}) + (\beta_{3,3}) + 2(\gamma_{3,4}).$$

Compare with Example 2.14.

The proof of Theorem 3.1 will occupy the rest of this section. Denote by e_i^T and f_i^T the Kashiwara operators on $\mathcal{T}(\infty)$ from Definition 2.5, and by e_i^{Kp} and f_i^{Kp} those on $\text{Kp}(\infty)$ from Definition 2.12.

Lemma 3.3. *Fix $i \in I$ and a row index j . Let $T \in \mathcal{T}(\infty)$ be such that the only unshaded boxes appearing in T occur in row j . Then $\text{br}_i(T)$ and $S_i(\Psi(T))$ have the same number of uncanceled brackets (both left and right). Furthermore, if $\text{br}_i(T)$ has an uncanceled left bracket, then $f_i^{\text{Kp}}\Psi(T) = \Psi(f_i^T T)$.*

Proof. First consider $i \in \{1, \dots, n-1\}$. We are only interested in entries $i, i+1, \overline{i+1}$, and \overline{i} , along with pairs $i-1, \overline{i-1}$, since these are the only entries that result in brackets in $\text{br}_i(T)$ or in $S_i(\Psi(T))$.

First consider a pair $i-1$ and $\overline{i-1}$: This corresponds to no brackets in $\text{br}_i(T)$, and to $(\beta_{j,i-1}), (\gamma_{j,i})$ in $\Psi(T)$, which gives a canceling pair of brackets in $S_i(\Psi(T))$. So the statement is true for T if and only if it is true for the tableau with this pair removed. Thus we can assume T has no such pairs.

Assume row j of T has p boxes of $\overline{i+1}$, q of $i+1$, r of i , s of \overline{i} :

$$R_j = \underbrace{\boxed{i} \cdots \boxed{i}}_r \underbrace{\boxed{i+1} \cdots \boxed{i+1}}_q \cdots \underbrace{\boxed{\overline{i+1}} \cdots \boxed{\overline{i+1}}}_p \underbrace{\boxed{\overline{i}} \cdots \boxed{\overline{i}}}_s.$$

We consider four cases.

Case 1: $p > q$ and $r > s$. Then

$$\Psi(R_j) = (r-s)(\beta_{j,i-1}) + s(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1})$$

and

$$f_i^T(R_j) = \underbrace{\boxed{i} \cdots \boxed{i}}_r \underbrace{\boxed{i+1} \cdots \boxed{i+1}}_q \cdots \underbrace{\boxed{\overline{i+1}} \cdots \boxed{\overline{i+1}}}_{p-1} \underbrace{\boxed{\overline{i}} \cdots \boxed{\overline{i}}}_{s+1}$$

giving

$$\begin{aligned}f_i^{\text{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1}) \\ &= \Psi(f_i^T R_j).\end{aligned}$$

Furthermore

$$\text{br}_i(R_j) =)^s ({}^p)^q ({}^r \quad \text{and} \quad S_i(\Psi(R_j)) =)^s ({}^{r-s} ({}^{s+p-q},$$

so both $\text{br}_i(R_j)$ and $S_i(\Psi(T))$ have s uncanceled ‘)’ and $r+p-q$ uncanceled ‘(.’

Case 2: $p > q$ and $r \leq s$. Then

$$\Psi(R_j) = r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r + p - q)(\gamma_{j,i+1}) + (s - r)(\gamma_{j,i})$$

and

$$f_i^T(R_j) = \underbrace{\boxed{i} \cdots \boxed{i}}_{r-1} \underbrace{\boxed{i+1} \cdots \boxed{i+1}}_{q+1} \cdots \underbrace{\boxed{\bar{i}+1} \cdots \boxed{\bar{i}+1}}_{p-1} \underbrace{\boxed{\bar{i}} \cdots \boxed{\bar{i}}}_{s+1}$$

giving

$$\begin{aligned} f_i^{\text{Kp}}\Psi(R_j) &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r + p - q - 1)(\gamma_{j,i+1}) + (s - r + 1)(\gamma_{j,i}) \\ &= \Psi(f_i^T R_j). \end{aligned}$$

Again, both $\text{br}_i(R_j)$ and $S_i(\Psi(T))$ have s uncanceled ‘)’ and $r + p - q$ uncanceled ‘(.’

Case 3: $p \leq q$ and $r > s$.

$$\Psi(R_j) = (r - s)(\beta_{j,i-1}) + (q - p + s)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1})$$

and

$$f_i^T(R_j) = \underbrace{\boxed{i} \cdots \boxed{i}}_{r-1} \underbrace{\boxed{i+1} \cdots \boxed{i+1}}_{q+1} \cdots \underbrace{\boxed{\bar{i}+1} \cdots \boxed{\bar{i}+1}}_p \underbrace{\boxed{\bar{i}} \cdots \boxed{\bar{i}}}_s$$

giving

$$\begin{aligned} f_i^{\text{Kp}}\Psi(R_j) &= (r - s - 1)(\beta_{j,i-1}) + (q - p + s + 1)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}) \\ &= \Psi(f_i^T R_j). \end{aligned}$$

Both $\text{br}_i(R_j)$ and $S_i(\Psi(T))$ have $s + q - p$ uncanceled ‘)’ and r uncanceled ‘(.’

Case 4: $p \leq q$ and $r \leq s$. Then

$$\Psi(R_j) = (q - p + r)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + r(\gamma_{j,i+1}) + (s - r)(\gamma_{j,i})$$

and

$$f_i^T(R_j) = \underbrace{\boxed{i} \cdots \boxed{i}}_{r-1} \underbrace{\boxed{i+1} \cdots \boxed{i+1}}_{q+1} \cdots \underbrace{\boxed{\bar{i}+1} \cdots \boxed{\bar{i}+1}}_p \underbrace{\boxed{\bar{i}} \cdots \boxed{\bar{i}}}_s$$

giving

$$\begin{aligned} f_i^{\text{Kp}}\Psi(R_j) &= (q - p + r)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + (r - 1)(\gamma_{j,i+1}) + (s - r + 1)(\gamma_{j,i}) \\ &= \Psi(f_i^T R_j). \end{aligned}$$

Again both $\text{br}_i(R_j)$ and $S_i(\Psi(T))$ have $s + q - p$ uncanceled ‘)’ and r uncanceled ‘(.’

The $i = n - 1$ case follows by the same argument, except there will never be both $i + 1$ and $\bar{i} + 1$ in the same row, so either p or q will be zero. The $i = n$ case follows from the $i = n - 1$ case using the Dynkin automorphism exchanging $n - 1$ and n , which has the effect on tableau of interchanging the symbols \bar{n} and n . (See Figure 2.1.) ■

Example 3.4. Consider type D_4 and $i = 2$, and the tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 4 & \bar{3} & \bar{1} & \bar{1} \\ \hline 2 & 2 & & & & & & & & \\ \hline 3 & & & & & & & & & \\ \hline \end{array}.$$

Then the reading word and bracketing sequence are

$$\text{br}_2(T) = \bar{1} \bar{1} \bar{3} 4 3 2 2 1 1 2 2 3$$

so

$$f_2^T T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 4 & \bar{3} & \bar{1} & \bar{1} \\ \hline 2 & 2 & & & & & & & & \\ \hline 3 & & & & & & & & & \\ \hline \end{array}$$

Direct calculation gives

$$\begin{aligned} \Psi(T) &= 2((\beta_{1,1}) + (\gamma_{1,2})) + ((\beta_{1,3}) + (\gamma_{1,4})) + 2(\beta_{1,1}) + (\beta_{1,3}) \\ &= 4(\beta_{1,1}) + 2(\beta_{1,3}) + (\gamma_{1,4}) + 2(\gamma_{1,2}) \end{aligned}$$

and

$$\begin{aligned} \Psi(f_2^T T) &= 2((\beta_{1,1}) + (\gamma_{1,2})) + ((\beta_{1,3}) + (\gamma_{1,4})) + (\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) \\ &= 3(\beta_{1,1}) + (\beta_{1,2}) + 2(\beta_{1,3}) + (\gamma_{1,4}) + 2(\gamma_{1,2}). \end{aligned}$$

The bracketing sequence on Kostant partitions is

$$S_2(\Psi(T)) = 0\beta_{1,2} \quad 4\beta_{1,1} \quad 2\gamma_{1,2} \quad 0\gamma_{1,3} \quad 0\beta_{2,2},$$

so $f_2^{\text{Kp}}(\Psi(T)) = \Psi(T) - (\beta_{1,1}) + (\beta_{1,1} + \alpha_2)$. Since $(\beta_{1,1} + \alpha_2) = (\beta_{1,2})$ this agrees with $\Psi(f_2(T))$.

Proof of Theorem 3.1. It suffices to show that, for all i , $f_i^T \Psi(T) = \Psi(f_i^{\text{Kp}} T)$. By the definition of the bracketing sequences and of Ψ we have

$$\text{br}_i(T) \text{ factors as } \text{br}_i(R_1)\text{br}_i(R_2) \cdots \text{br}_i(R_{n-1}), \text{ and}$$

$$S_i(\Psi(T)) \text{ factors as } S_i(\Psi(R_1))S_i(\Psi(R_2)) \cdots S_i(\Psi(R_{n-1})).$$

By Lemma 3.3, each $\text{br}_i(R_t)$ has the same number of uncanceled brackets as each $S_i(\Psi(R_t))$. Hence the first uncanceled '(' in $\text{br}_i(T)$ and in $S_i(\Psi(T))$ occur in the same factor, say from row R_j . But then, also by Lemma 3.3, $f_i^T \Psi(R_j) = \Psi(f_i^{\text{Kp}} R_j)$, so in fact $f_i^T \Psi(T) = \Psi(f_i^{\text{Kp}} T)$. ■

4. STACK NOTATION

As mentioned in the introduction, this work is a type D analogue of a type A result found in [3]. That type A result may be described within the framework of multisegments [7, 9, 13], which have the advantage of a convenient diagrammatic notation which makes the crystal structure apparent. By analogy, one may introduce a *stack* notation for Kostant partitions in type D in which the crystal structure may be read off easily.

REFERENCES

- [1] Arkady Berenstein and Andrei Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no. 1, 77–128, [arXiv:math/9912012](https://arxiv.org/abs/math/9912012).
- [2] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.
- [3] John Claxton and Peter Tingley, *Young tableaux, multisegments, and PBW bases*, Sémin. Lothar. Combin. **73** (2015), Article B73c, [arXiv:1503.08194](https://arxiv.org/abs/1503.08194).
- [4] The Sage Developers, *Sage Mathematics Software (Version 7.2)*, 2016, <http://www.sagemath.org>.
- [5] Jin Hong and Seok-Jin Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002.
- [6] Jin Hong and Hyeonmi Lee, *Young tableaux and crystal $\mathcal{B}(\infty)$ for finite simple Lie algebras*, J. Algebra **320** (2008), no. 10, 3680–3693, [arXiv:math/0507448](https://arxiv.org/abs/math/0507448).
- [7] Nicolas Jacon and Cédric Lecouvey, *Kashiwara and Zelevinsky involutions in affine type A*, Pacific J. Math. **243** (2009), no. 2, 287–311, [arXiv:0901.0443](https://arxiv.org/abs/0901.0443).
- [8] Masaki Kashiwara, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.
- [9] Bernard Leclerc, Jean-Yves Thibon, and Eric Vasserot, *Zelevinsky’s involution at roots of unity*, J. Reine Angew. Math. **513** (1999), 33–51, [arXiv:math/9806060](https://arxiv.org/abs/math/9806060).
- [10] George Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010, Reprint of the 1994 edition.
- [11] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, 2016, <http://combinat.sagemath.org>.
- [12] Ben Salisbury, Adam Schultze, and Peter Tingley, *Combinatorial descriptions of the crystal structure on certain PBW bases*, [arXiv:1606.01978](https://arxiv.org/abs/1606.01978).
- [13] A. V. Zelevinsky, *Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$* , Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210.

DEPARTMENT OF MATHEMATICS, CENTRAL MICHIGAN UNIVERSITY, MOUNT PLEASANT, MI
E-mail address: ben.salisbury@cmich.edu
URL: <http://people.cst.cmich.edu/salis1bt/>

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY AT ALBANY, ALBANY, NY
E-mail address: alschultze@albany.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, LOYOLA UNIVERSITY, CHICAGO, IL
E-mail address: ptingley@luc.edu
URL: <http://webpages.math.luc.edu/~ptingley/>